Heisenberg limited measurements with superconducting circuits

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We describe an assembly of N superconducting qubits contained in a single-mode cavity. In the dispersive regime, the correlation between the cavity field and each qubit results in an effective interaction between qubits that can be used to dynamically generate maximally entangled states. With only collective manipulations, we show how to create maximally entangled quantum states and how to use these states to reach the Heisenberg limit in the determination of the qubit bias control parameter (gate charge for charge qubits, external magnetic flux for rf-SQUIDs).

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The description of the interaction between atoms and quantized modes of the electromagnetic field in a cavity is called cavity quantum electrodynamics (cQED). The first experimental studies used flying Rydberg atoms in a rf resonator [1]. With the advent of quantum computing several other implementations were developed to mimic the quantum properties of atoms. Among those, solid-state implementations are especially interesting since they offer several advantages over real atoms: these artificial atom properties can be tailored and their number and location are fixed. We concentrate our discussion on superconducting circuits. Superconducting implementations of quantum computing attracted a lot of attention in recent years because they are inherently scalable and single qubit operations have been demonstrated with classical coherent control in a variety of qubit types. Several theoretical studies have treated the interaction of superconducting qubit(s) with a quantized electromagnetic field. The proposal of an on-chip cQED experiment using a Cooper-pair box as the artificial atom strongly coupled to a one-dimensional cavity [2] is especially interesting since it was followed by several experiments. First, the strength of the coupling was shown to be indeed stronger than the different decay constants so that the vacuum Rabi splitting was observed [3]. Subsequently the ac-Stark shift was measured using a quantum non-demolition technique (QND) [4] and the decoherence time T_2 evaluated from Ramsey fringe experiments [5].

In a cavity with a high quality factor, the photons can serve as an information bus between several qubits and therefore create correlations between distant qubits. Beside its fundamental interest and applications to quantum information processing, entanglement offers the additional advantage of allowing improved sensitivity in a quantum-limited measurement. Compared to a measurement made with only one sensor, the sensitivity is improved by $1/\sqrt{N}$ when N classical sensors are used to measure the same quantity. Now, if N quantum sensors are *coherently* coupled, the sensitivity improvement can reach $1/N$. This ultimate sensitivity is referred to as the Heisenberg limit. Techniques to create and control multiparticle entangled states necessary to reach this limit are already available in different implementations. For instance, interferences between two different polarizations (modes) of three and four photon entangled states have been observed [6, 7] and an experiment performed on an assembly of three beryllium ions demonstrated a spectroscopic sensitivity improvement [8]. Today's solid-state sensors do not take advantage of the improvement made possible by quantum correlations. Using entanglement in a superconducting implementation to enable such Heisenberg limited measurements, could therefore revolutionize sensor technology with, for instance, electrometers and magnetometers.

In this work, we use techniques from other fields (atomic QED, ion traps, quantum optics) to describe the interactions and manipulations needed in a superconducting circuit to perform a Heisenberg limited measurement. We propose to use the photon-qubit interaction to create an effective interaction between distant superconducting qubits. This interaction is used to generate maximally entangled states which in turn are used to beat the standard quantum limit when measuring the natural frequency of the system. We show how this results in a Heisenberg limited estimation of the qubit bias control parameter.

Most superconducting qubits can be described by the following single qubit Hamiltonian [9]:

$$H_Q = -\frac{B_z}{2}\,\sigma_z - \frac{B_x}{2}\,\sigma_x, \tag{1}$$

with the bias B_z depending linearly on the dimensionless control parameter λ: $B_z = b_z(1/2 - \lambda)$. This Hamiltonian is an approximation valid around a symmetry point, obtained for $\lambda = 1/2$, called the degeneracy point. The parameters B_z, B_x, b_z and λ for the two main types of superconducting qubits, can be found in the review article [9] on Josephson based devices. The bias is controlled with an electric gate charge n_g in a Cooper-pair box (CPB) or with an external magnetic flux ϕ_x in a rf-SQUID.

We now consider that the qubit is contained in a cavity described by the Hamiltonian $H_c = \omega_c(a^\dagger a + 1/2)$. The quantized cavity mode adds an incremental contribution $\delta\lambda$ to the bias so that $\lambda + \delta\lambda = \lambda + \lambda_c(a^\dagger + a)$. The single qubit Hamiltonian now reads:

$$H_{Q-C} = -\frac{B_z}{2}\sigma_z - \frac{B_x}{2}\sigma_x + \frac{b_z\lambda_c}{2}(a^\dagger + a)\sigma_z. \quad (2)$$

We assume the strong coupling limit $g \gg \kappa, \gamma$. We neglect the cavity decay κ and excited state decay γ for the moment. Experiments on a single CPB in a cavity [3] support this assumption ($g \approx 10\,\kappa$). We rewrite this Hamiltonian in the eigenbasis of (1) which is done by performing a rotation of θ about the y axis. The mixing angle θ is given by $\tan\theta = B_x/B_z$. The resulting individual contribution is then summed N times to describe an assembly of N identical qubits in a single mode cavity:

$$\begin{aligned} H_N &= \Omega\, S_z + \omega_c(a^\dagger a + 1/2) \\ &+ 2g(a^\dagger + a)(S_z\cos\theta + S_x\sin\theta), \end{aligned} \quad (3)$$

where $2g = -b_z\lambda_c$, $\Omega = \sqrt{B_x^2 + B_z^2}$ and $2\vec{S} = \sum_{i=1}^N \vec{\sigma_i}$. Of course, the assumption of exact parameter degeneracy is rather stringent. Our intent here is to study the ideal case to obtain simple analytic expressions. When $\theta = \pi/2$ and the rotating wave approximation is made, equation (3) takes the form of the N-particle Jaynes-Cummings Hamiltonian. When $\theta = 0$, equation (3) describes a harmonic oscillator with a conditional displacement, *e.g.* a displacement that depends on the total state of the system S_z. However, the limit $\theta \to 0$ is reached when the bias B_z is maximum and $B_x \to 0$. In this case, the two-level approximation (1), and hence equation (3), is not justified. We assume that the qubit-cavity detuning $\Delta = \Omega - \omega_c$ is much larger than the qubit coupling g to the cavity. We write $S_x = (S_+ + S_-)/2$ and neglect the rapidly oscillating terms. Hamiltonian (3) can be approximately diagonalized with a polaronic transformation $U = \exp\left(\frac{g\sin\theta}{\Delta}(aS_+ - a^\dagger S_-)\right)$:

$$\begin{aligned} \tilde{H}_N = UH_NU^\dagger &\approx \Omega\, S_z + \omega_c(a^\dagger a + 1/2) \\ &+ \chi(S^2 - S_z^2 + S_z + 2a^\dagger a S_z). \end{aligned} \quad (4)$$

We define $\chi = (g\sin\theta)^2/\Delta$. A more complete diagonalization would require to also take into account the displaced harmonic oscillator with $U_d = \exp(\frac{2g\cos\theta}{\omega_c}(a - a^\dagger)S_z)$ and perform the transformation $U_dUH_NU^\dagger U_d^\dagger$. However, this complete transformation introduces terms proportional to g^2/ω_c which we neglect because we are interested in the regime where $g \ll \Delta \ll \omega_c$. We also neglect terms in χ^2 that would appear with the complete transformation. Hence, equation (4) is valid to first order in χ. The coupling of the qubits with the resonator induces a shift in the qubit frequency and a state dependent shift in the resonator frequency. These Lamb and ac-Stark shifts (respectively) were predicted in the case of a single CPB in a cavity [2].

The novelty of equation (4) compared to previously published results is the appearance of an effective interaction $H_{sz} = \chi S_z^2$ between N qubits mediated by a cavity photon. This interaction is also known as a *one-axis twisting* interaction [10] because it twists around the z axis the quantum fluctuations of the total spin $\vec{S}$ (for a system with several spins). Because of this feature, the interaction H_{sz} is a key element in non-optical implementations of Heisenberg limited estimations. It has been used extensively in ion traps experiments to generate maximally entangled states [11, 12, 13]. It has been shown to be possible to utilize these states to perform an optimal frequency measurement [14] and improve the estimation of rotation angles [15]. More recently, a method involving only collective manipulations has been used to perform precision spectroscopy on an assembly of six beryllium ions [16]. The first step consists in generating the maximally entangled state $|\psi_m\rangle$ (see for instance reference [17]) using the time evolution of H_{sz} over a time $t_{sz} = \pi/2\chi$:

$$\begin{aligned} |\psi_m\rangle &= e^{-i\frac{\pi}{2}S_z^2}|-N/2\rangle_x = \\ &\frac{1}{\sqrt{2}}\Big(|-N/2\rangle_x + i^{N+E}|+N/2\rangle_x\Big). \end{aligned} \quad (5)$$

When the number of qubit N is odd, another rotation $e^{i\frac{\pi}{2}S_z}$ is needed in addition to the $e^{i\frac{\pi}{2}S_z^2}$ [17]. We take the parity into account in equation (5) by setting the quantity E to 2 (1) when N is odd (even resp.). Initially all the qubits are in their ground state, so that the total wave function describing all the qubits is $|J = N/2, M = -N/2\rangle_z \equiv |-N/2\rangle_z = |\downarrow\rangle_1|\downarrow\rangle_2\cdots|\downarrow\rangle_N$ (we do not consider the field part of the wave function at this point). In order to prepare a quantum state according to equation (5), we need to rotate the initial state $|-N/2\rangle_z$ around the x axis. Therefore, we define the operator $U_N = e^{i\frac{\pi}{2}S_x}e^{-iH_{sz}t_{sz}}e^{-i\frac{\pi}{2}S_x}$ for notation convenience. The average of the total spin vector calculated with the wave function $|\psi_m\rangle$ is zero, $\langle\vec{S}\rangle = 0$. Thus, the natural choice of S_z ($\vec{S}$) as the observable to be measured can not be made. Bollinger *et al.* showed that the parity operator $\prod_{i=1}^N \sigma_{z_i}$ was an adequate observable that could be measured with the state (5). However the measurement of this operator for a large number N of qubits is difficult since it requires distinguishing odd and even numbers of particles in state $|\downarrow\rangle$. A method involving only collective manipulations has been proposed in [8] to circumvent this problem. First the maximally entangled state is constructed. In our case the sequence U_N is applied to the initial state $|-N/2\rangle_z$. Afterward, the system evolves freely during a period T, obeying the dynamics defined by the Hamiltonian $\tilde{H}_N$. Finally, another application of the sequence U_N transfers the phase information, $N\phi/2$, into an amplitude information of either state $|+N/2\rangle_z$ or $|-N/2\rangle_z$:

$$\begin{aligned} |\psi\rangle = U_N e^{-i\tilde{H}_N T} U_N |-N/2\rangle_z = \\ -i \sin\left(\frac{N}{2}\,\phi\right)|-N/2\rangle_z + \\ i^{N+E} \cos\left(\frac{N}{2}\,\phi\right)|+N/2\rangle_z. \end{aligned} \tag{6}$$

A measurement will collapse the wave function $|\psi\rangle$ on either state $|+N/2\rangle_z$ or state $|-N/2\rangle_z$ with probability $P_\uparrow = \frac{1}{2}(1+\cos(N\phi))$ or $P_\downarrow = \frac{1}{2}(1-\cos(N\phi))$.

We propose to use cavity spectroscopy to infer the state of the qubits. Assuming that there is a finite but small cavity decay rate κ, a signal at frequency ω_c sent in the cavity will experience a phase shift when it is transmitted. Solving the Heisenberg equation for the field creation operator, this phase shift ϑ is given by $\tan\vartheta = \pm(2\chi N)/\kappa$. The probability $P_\uparrow$ is extracted from the time dependence of ϑ. A measurement scheme as been proposed based on this principle to perform a quantum non-demolition measurement of the state of a single Cooper-pair box contained in a cavity [2]. The difference is that because the coupling to the cavity is $\sqrt{N}$ stronger, the phase shift ϑ is larger than in the single qubit case.

The main motivation to use N-particle maximally entangled state to perform a spectroscopy measurement is to be able to relate the N fold frequency increase to the phase uncertainty. The uncertainty on a parameter ζ can be estimated from the error propagation formula $\delta\zeta = \Delta\hat{A}/|\partial\langle\hat{A}\rangle/\partial\zeta|$ by measuring the operator $\hat{A}$. We introduce the projection operator $\hat{A} = |+N/2\rangle\langle+N/2|$ so that the quantity we propose to measure $P_\uparrow$ is the average of $\hat{A}$ over the state $|\psi\rangle$ of equation (6). The variance $\Delta\hat{A}^2$ is then simply given by $P_\uparrow(1-P_\uparrow)$ (second moment of the Bernoulli distribution) which is equal to $(\sin(N\phi)/2)^2$. The denominator of the error propagation formula, $|\partial P_\uparrow/\partial\phi|$, is $N \times \sin(N\phi)/2$. Hence, the measurement of $P_\uparrow$ leads to an estimation of the phase uncertainty $\delta\phi$ equal to $1/N$. The phase acquired during the free evolution, by the qubit part of the wave function, is $(\Omega + \chi + 2\chi\bar{n}) \times T$ where $\bar{n}$ is the average photon number in the cavity. The contribution of the frequency shifts is negligible in the expression of the phase uncertainty. Thus, the frequency uncertainty is given by $\delta\Omega = 1/(NT)$. In superconducting circuits, the parameter λ controls the level spacing Ω and therefore the uncertainties of both quantities can be related through the following relation:

$$\delta\Omega = \frac{b_z|B_z|}{\sqrt{B_x^2 + B_z^2}}\,\delta\lambda = b_z|\cos\theta|\,\delta\lambda \tag{7}$$

Hence, a measurement of the frequency Ω performed with the sequence of operations defined by equation (6) results in a Heisenberg limited measurement of the parameter λ, *e.g.* in an improvement of the uncertainty $\delta\lambda = 1/(NT \times b_z|\cos\theta|)$ associated with the estimation of λ. This method can be used to improve the estimation of the gate charge n_g (or the external bias flux ϕ_x) in a system composed of N Cooper-pair boxes (or rf-SQUIDs resp.) that are coherently coupled. The energies b_z and B_x can be determined from a preliminary spectroscopy experiment. Sweeping λ when applying a periodic signal will flip the qubits when the frequency is resonant.

Our scheme is useful *away* from the degeneracy point as at this point $\cos\theta = 0$. However, one should bear in mind that the coupling χ decreases as the operating point is moved away from the degeneracy point, so a trade-off should be made to operate away, while not too far from this point. The limited validity range around the degeneracy point is not a limitation in a system composed of N Cooper-pair boxes since the different coherence times decrease with the distance from this point [18] and therefore the operation far away from it, is not adequate to observe coherent effects.

To aid the understanding of our scheme in particular and of Heisenberg limited measurements in general, we wish to emphasize the difference between the quantum observable measured and the parameter that the method allows to determine with a better precision. The quantum observable defines the type of superconducting sensors or qubit. To simplify the discussion, let's say there are mainly two types of superconducting devices, electric charge and magnetic flux sensors. The quantum observable measured is then either the electric charge $\hat{n}$ or the magnetic flux $\hat{\phi}$. Now, the Hamiltonian of the system can be tuned or controled with a classical (continuous) variable which is the gate charge n_g in a charge qubit or the external magnetic flux ϕ_x in a flux qubit. A Heisenberg limited measurement consists of evaluating the uncertainty of the parameter (δn_g or $\delta\phi_x$) for a given value of this parameter with a measurement of the quantum observable ($\hat{n}$ or $\hat{\phi}$). The parameters usually estimated in a Heisenberg limited measurement are either an energy splitting, a rotation angle or a phase delay. Therefore, by establishing for the first time a relation between other parameters, such as the electric charge or the magnetic flux, and a quantum measurement, we show how a Heisenberg limited measurement can have some applications in sensor technology.

In this work, we described a collection of N superconducting qubits contained in a single mode cavity. Besides the usual shifts in the qubit and resonator frequencies, we find that the effective Hamiltonian contains a term χS_z^2 describing the interaction between all the qubits. This interaction can be used to dynamically prepare maximally entangled states. We adapt a method used in ion traps to demonstrate the use of these states to reach the Heisenberg limit in the determination of the qubit frequency. Finally, we show that the parameter that controls the energy spacing can be estimated with an uncertainty that scales inversely with the number N of qubits. Hence our work establishes the first formal relation between, either the electric charge or the magnetic flux in an assembly of superconducting devices, and the so-called Heisenberg limited fluctuations.

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